Preprint

Squeezing of Atomic Quantum Projection Noise

Patrick J. Windpassinger, Daniel Oblak, Ulrich B. Hoff,
Anne Louchet, Jürgen Appel, Niels Kjærgaard* and Eugene S. Polzik

Danish National Center for Quantum Optics—QUANTOP,
Niels Bohr Institute, University of Copenhagen, Denmark

(*March 24, 2009*)

We provide a framework for understanding recent experiments on squeezing of a collective atomic pseudo-spin, induced by a homodyne measurement on off-resonant probe light interrogating the atoms. The detection of light decimates the atomic state distribution and we discuss the conditions under which the resulting reduced quantum fluctuations are metrologically relevant. In particular, we consider a dual probe scheme which benefits from a cancelation of classical common mode noise sources such that quantum fluctuations from light and atoms are the main contributions to the detected signal.

Keywords: projection noise; quantum noise squeezing; quantum non-demolition measurement

1. Introduction

During the past year, experiments focusing on pseudo-spin squeezing via quantum non-demolition (QND) measurements (*1*) have received considerable interest (*2, 3, 4, 5, 6*). As reported in (*6*), our group recently demonstrated 3.4 dB of spectroscopically relevant quantum noise squeezing for up to $\sim 10^5$ caesium atoms via a QND measurement of the population difference between the clock levels. The QND measurement was implemented by interferometric detection of the state dependent phase shift of an off-resonant dichromatic probe light pulse. In the present paper we shall illuminate some of the underlying physics in play for such a measurement induced squeezing protocol.

2. Atomic projection noise

Imagine N two-level atoms with energy eigenstates $|{\downarrow}\rangle$ and $|{\uparrow}\rangle$. With all atoms initially in the $|{\downarrow}\rangle$ state a resonant $\pi/2$-pulse is applied on the $|{\downarrow}\rangle \leftrightarrow |{\uparrow}\rangle$ transition preparing each atom in the superposition state $(|{\downarrow}\rangle + \mathrm{i}\,|{\uparrow}\rangle)/\sqrt{2}$ so that the collective atomic state is described by

$$|\Psi\rangle_{\mathrm{CSS}} = \bigotimes_{k=1}^{N} (|{\downarrow}\rangle + \mathrm{i}\,|{\uparrow}\rangle)_k/\sqrt{2}. \tag{1}$$

A projective measurement of the number of atoms in the $|{\uparrow}\rangle$-state is connected to an expectation value of $\langle N_\uparrow\rangle = N/2$ and a variance $\mathrm{var}(N_\uparrow) = N/4$. It follows that measurements of the

*Corresponding author. Email: niels.kjaergaard@gmail.com

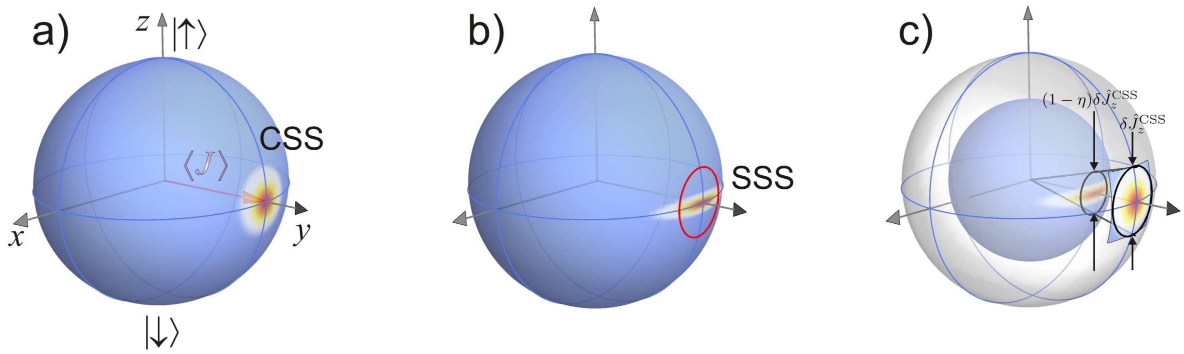

Figure 1. a) Bloch sphere representation of the coherent spin state (CSS) of Eq. (1). The pointing uncertainty of the Bloch vector with mean $\langle \hat{\mathbf{J}} \rangle$ in pseudo-spin space is illustrated as a shaded disk. b) Illustration of a spin squeezed state (SSS) with reduced fluctuations of the $\hat{J}_z$ projection as compared to the CSS. c) If the protocol for obtaining an SSS from a CSS is accompanied by decoherence which reduces the radius of the original Bloch sphere by a factor $(1-\eta)$ the fluctuations of the SSS $\delta\hat{J}_z^{\mathrm{SSS}}$ should be less than $(1-\eta)\delta\hat{J}_z^{\mathrm{CSS}}$ in order for the squeezed component to increase angular resolution in yz-plane (spectroscopically relevant squeezing).

population *difference* between $|\downarrow\rangle$ and $|\uparrow\rangle$ will fluctuate about a zero mean with a variance $\mathrm{var}(N_\uparrow - N_\downarrow) = \mathrm{var}(2N_\uparrow - N) = 4\mathrm{var}(N_\uparrow) = N$. These fluctuations are referred to as atomic projection noise and and they restrict the precision to which we can retrieve the value $\pi/2$ for the original angle of rotation from $|\downarrow\rangle^{\otimes N}$ based on a measurement of the $|\downarrow\rangle$ and $|\uparrow\rangle$ population difference. Introducing collective atomic operators

$$\hat{J}_x = \frac{1}{2}\sum_{k=1}^{N}(|\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|)_k, \tag{2a}$$

$$\hat{J}_y = \frac{-\mathrm{i}}{2}\sum_{k=1}^{N}(|\uparrow\rangle\langle\downarrow| - |\downarrow\rangle\langle\uparrow|)_k, \tag{2b}$$

$$\hat{J}_z = \frac{1}{2}\sum_{k=1}^{N}(|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|)_k, \tag{2c}$$

which may be shown to obey angular momentum commutation relations $[\hat{J}_k, \hat{J}_l] = i\epsilon_{klm}\hat{J}_m$, we can rewrite $|\Psi\rangle_{\mathrm{CSS}}$ in terms of simultaneous eigenstates of $\hat{J}_z$ and $\hat{\mathbf{J}}^2$

$$|\Psi\rangle_{\mathrm{CSS}} = \frac{1}{2^{N/2}}\sum_{M=-N/2}^{N/2}\binom{N}{N/2+M}^{1/2}|N/2, M\rangle, \tag{3}$$

i.e., the (sub-)set of Dicke states (*7, 8*) fulfilling $\hat{J}_z|N/2,M\rangle = M|N/2,M\rangle$ and $\hat{\mathbf{J}}^2|N/2,M\rangle = (N/2)(N/2+1)|N/2,M\rangle$. $|\Psi\rangle_{\mathrm{CSS}}$ can be obtained by a rotation $(\theta = \pi/2, \varphi = 0)$ of the Dicke state $|N/2,-N/2\rangle$ in angular momentum space and is referred to as a Bloch state, an atomic coherent state, or a coherent spin state (CSS). The polar angle θ and azimuth φ parameterize the Bloch sphere of radius $N/2$. As illustrated in Fig. 1(a), we can represent the uncertainty associated with a projective measurement on $|\Psi\rangle_{\mathrm{CSS}}$ as a disk of radius $\sqrt{N}$ on the Bloch sphere, where the uncertainty disk is centered on the tip of the Bloch vector $\langle\hat{\mathbf{J}}\rangle$. We conclude that Ramsey spectroscopy which essentially relies on determining such angular rotations of a Bloch vector is ultimately limited by atomic projection noise (*9*). The prospect of surpassing this so-called standard quantum limit (SQL) motivates work towards the production of squeezed atomic states [see Fig. 1(b)] displaying a reduction of quantum projection noise in a given

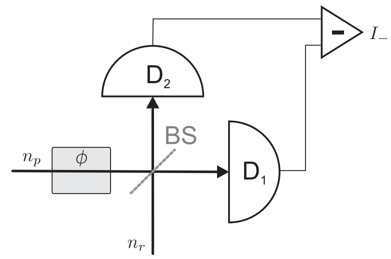

Figure 2. Balanced homodyne measurement of the phase shift ϕ between a probe and reference light field as the difference signal I_- between photo detectors D1 and D2.

measurement basis (*10*).

3. Projection noise measurements and squeezing

In order to infer squeezing of atomic quantum noise it is obviously a requirement to have a measurement scheme with sufficient sensitivity to reveal the projection noise limit for a CSS in the first instance. However, if we have the ability to perform such a projection noise limited measurement on a CSS and, moreover, the measurement has a QND character we have already succeeded in producing a squeezed atomic state. The very act of measuring defines one component of the collective pseudo-spin (say, the population difference) to a precision better than the SQL for a CSS. Such a QND measurement can be implemented by determining the refractive index of the atomic ensemble using off-resonant probe light (*1*).

3.1. *Refractive index measurements via phase shift of light*

When a probe beam of light of wavelength λ propagates a distance l through a medium of refractive index n it acquires a phase shift $\phi = \frac{2\pi l}{\lambda}(n-1)$. The phase shift can be measured in a balanced homodyne detection setup (see Fig. 2) where in essence the phase of the probe beam is compared to the phase of a reference beam (local oscillator). Describing the probe and reference pulses of light by coherent states with average photon numbers n_p and n_r, respectively, the differential signal from the two detectors yields (on average) a signal (see appendix A)

$$\langle I_- \rangle = 2\sqrt{n_r n_p}\sin\phi. \tag{4}$$

with a second moment given by

$$\langle I_-^2 \rangle = n_r + n_p + 4n_r n_p \sin^2\phi. \tag{5}$$

For a *constant* phase shift the differential detector signal fluctuates as $\mathrm{var}\,(I_-) = \langle I_-^2\rangle - \langle I_-\rangle^2 = n_r + n_p$ independent of ϕ which is nothing but the shot noise of light. For a fluctuating phase ϕ we obtain using the law of total variance

$$\begin{aligned}\mathrm{var}\,(I_-) &= \langle \mathrm{var}\,(I_-|\phi)\rangle + \mathrm{var}\,(\langle I_-|\phi\rangle)\\ &= n_r + n_p + \mathrm{var}\left(2\sqrt{n_r n_p}\sin\phi\right) \approx n_r + n_p + \mathrm{var}\left(2\sqrt{n_r n_p}\phi\right)\\ &= n_r + n_p + 4n_r n_p \mathrm{var}(\phi),\end{aligned} \tag{6}$$

where we have assumed that $\phi \ll \pi/2$ so that $\sin\phi \approx \phi$.

3.1.1. Projection noise measurements on a CSS using one and two probes

Suppose now that the refractive medium is made up by N atoms in a CSS as described in section 2 and moreover that the probe laser phase shift is proportional to the number $N_\uparrow$ of atoms in the $|\uparrow\rangle$-state:

$$\phi_\uparrow = k_\uparrow N_\uparrow + \phi_0, \tag{7}$$

where ϕ_0 is a fluctuating background phase not related to the atoms. By off-setting the phase of the reference beam by $\langle\phi_0\rangle + \langle k_\uparrow\rangle N/2$, the differential signal I_- (representing the phase difference between the reference and probe fields) fluctuates about zero and the approximation $\sin\phi \approx \phi$ is valid. Equation (6) then yields (cf. appendix B.1)

$$\mathrm{var}\left(I_-^{(\uparrow)}\right) = n_r + n_p + n_r n_p \left\{\left[\langle k_\uparrow\rangle^2 + \mathrm{var}\left(k_\uparrow\right) N + \mathrm{var}\left(k_\uparrow\right)\right] N + 4\mathrm{var}\left(\phi_0\right)\right\}. \tag{8}$$

We next introduce an additional probe laser with mean photon number m_p sensitive to the population $N_\downarrow$ of the $|\downarrow\rangle$ state. This probe is homodyned with a reference (mean photon number m_r) having a local oscillator phase such that

$$\phi_\downarrow = -(k_\downarrow N_\downarrow + \phi_0). \tag{9}$$

For $k_\downarrow = k_\uparrow \equiv k$ and $n_r n_p = m_r m_p$ we get (cf. appendix B.2)

$$\begin{aligned}\mathrm{var}\left(I_-^{(\uparrow\downarrow)}\right) &= n_r + n_p + m_r + m_p + 4n_r n_p\left[\langle k\rangle^2 + \mathrm{var}(k)\right] N \\ &\approx \underbrace{n_r + n_p + m_r + m_p}_{\text{shot noise}} + \underbrace{4n_r n_p \langle k\rangle^2 N}_{\text{projection noise}},\end{aligned} \tag{10}$$

where the approximation in the second line is valid if $\mathrm{var}(k) \ll \langle k\rangle^2$. The variance of the differential detector signal for dual probe interrogation Eq. (10) displays a shot noise term and a projection noise term, the latter scaling linearly with the atom number N. In contrast to single probe interrogation Eq. (8) there is a cancelation of background phase fluctuations not relating to atoms as well as the component quadratic in the number of atoms.

3.1.2. Two-color dispersive probing scheme

In practice, to achieve a dual probe interrogation scheme (*11*) we can consider a situation as shown in Fig. 3. Here the frequency of the probe $P_\uparrow(P_\downarrow)$ is detuned $\Delta_\uparrow(\Delta_\downarrow)$ from resonance of the closed optical transition between the $|\uparrow\rangle$ ($|\downarrow\rangle$) state and an auxiliary quantum state $|1\rangle$ ($|2\rangle$). If the detunings are large compared to the respective line widths $\Gamma_\uparrow(\Gamma_\downarrow)$ of these two transitions the dispersive interaction responsible for the phase shift of light from atoms will have a simple $1/\Delta_{\uparrow(\downarrow)}$ dependence, i.e.:

$$k_\downarrow = \frac{c_{\downarrow,1}}{\Delta_\downarrow} \quad , \quad k_\uparrow = \frac{c_{\uparrow,2}}{\Delta_\uparrow}, \tag{11}$$

where $c_{\downarrow,1}$ and $c_{\uparrow,2}$ are constants comprising atomic parameters and the probing geometry. For comparable detunings $\Delta_\uparrow \approx \Delta_\downarrow$ the task of keeping $k_\uparrow = k_\downarrow$ translates into stabilizing the difference in frequency between the two probe fields which is experimentally feasible (*12*) when the transition $|\downarrow\rangle \leftrightarrow |\uparrow\rangle$ is in the radio frequency domain (which is the case for hyperfine split states). The requirement $\mathrm{var}(k) \ll \langle k\rangle^2$ is then fulfilled if the absolute frequency fluctuations of $P_\uparrow$ is $\ll \Delta_\uparrow$ which is also comfortably met for $\Delta \gtrsim 100$ MHz, e.g. by referencing $P_\uparrow$ to an atomic transition. In contrast, single probe interrogation poses a criterion $\mathrm{var}(k) \ll \langle k\rangle^2/N$

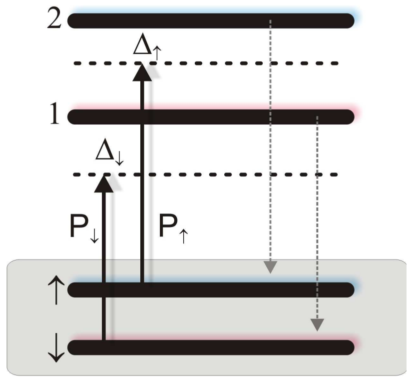

Figure 3. Level diagram for a two color dispersive probing scheme. The populations of the levels $\downarrow$ and $\uparrow$ are measured with optical probes which interacts off-resonantly with the auxiliary levels 1 and 2, respectively.

which, e.g., requires the probe frequency fluctuations to be $\ll \Delta_\uparrow/\sqrt{N}$. This quickly becomes a serious challenge for ensembles containing hundred thousands of atoms.

3.1.3. Measurement based quantum noise reduction

When no atoms are present the differential detector signal will fluctuate about zero with a variance given by the shot noise term in Eq. (10) $n_{\mathrm{sn}} \equiv n_r + n_p + m_r + m_p$. If we introduce atoms in a Dicke state $|N/2, M\rangle$, i.e. a *fixed* atomic phase shift, the I_- reading would continue to fluctuate with a variance n_{sn}, but now about a mean value $4\sqrt{n_r n_p}kM \equiv gM$. The probability distribution of the photonic signal I_- conditioned on the atomic state $|N/2, M\rangle$ is then

$$P\left(I_- = n \middle| \, |N/2, M\rangle\right) = \frac{1}{\sqrt{2\pi n_{\mathrm{sn}}}} \exp\left[-\frac{(n - gM)^2}{2n_{\mathrm{sn}}}\right], \tag{12}$$

where the Skellam distributed differential photon current has been approximated by a Gaussian. With the atoms prepared in the CSS Eq. (3) which is a distribution over Dicke states, the probability for a given Dicke state is

$$P(|N/2, M\rangle) = |\langle N/2, M| \, \Psi\rangle_{\mathrm{CSS}}|^2 = \frac{1}{2^N}\binom{N}{N/2 + M}. \tag{13}$$

Using Bayes' rule the distribution over M-states given a detection event $I_- = n$ can be inferred

$$\begin{aligned} P\left(|N/2, M\rangle \, \middle| I_- = n\right) &\propto \exp\left[-\frac{(n - gM)^2}{2n_{\mathrm{sn}}}\right] \exp\left[-\frac{M^2}{N/2}\right] \\ &\propto \exp\left[-\frac{\left(M - \frac{gN/4n_{\mathrm{sn}}}{1+g^2N/4n_{\mathrm{sn}}} n\right)^2}{\frac{N/2}{1+g^2N/4n_{\mathrm{sn}}}}\right]. \end{aligned} \tag{14}$$

Compared to the initial CSS $|\Psi\rangle_{\mathrm{CSS}}$ this $\hat{J}_z$ projection distribution has a variance reduced by a factor $1 + g^2N/4n_{\mathrm{sn}} \equiv 1 + \kappa^2$ and is centered at the value $n\kappa^2/g(1 + \kappa^2)$. We refer to this as a conditionally squeezed state since based on the measurement outcome $I_- = n$ we can predict the outcome of a subsequent $\hat{J}_z$ measurement to a precision better than the SQL for a CSS. In Fig. 4 we illustrate the process of quantum noise reduction (and the shift of the projection distribution) as a result of a particular measurement outcome of I_-. The parameter $\kappa^2 \equiv g^2N/4n_{\mathrm{sn}} = 4n_r n_p k^2 N/(n_r + n_p + m_r + m_p)$ characterizes the amount of squeezing that is achieved. From Eq. (10) we see that this is nothing but the ratio of atomic projection noise to the shot noise of light encountered in the signal I_-. In the case of strong local oscillators of

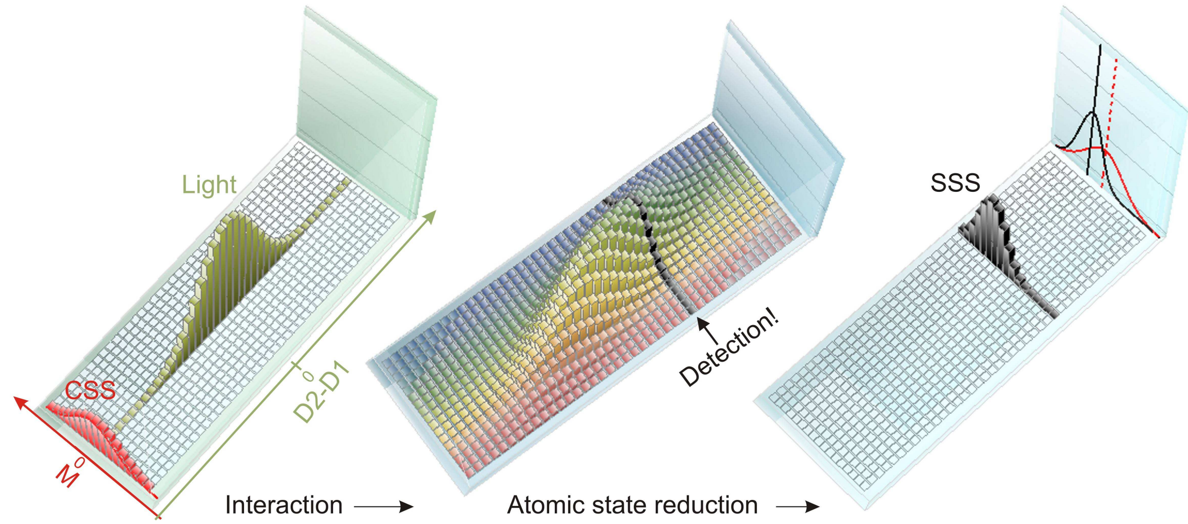

Figure 4. Graphical illustration of atomic projection noise reduction based upon the state reduction which happens after the detection of phase shifted probe light. The interaction between light and a CSS (left pane) gives rise to the joint distribution in the center pane. Upon detection of a differential photo detector signal $I_- = D2 - D1$ the atomic distribution over M-states is decimated as illustrated in the right pane. This state has $\hat{J}_z$-projection which is defined beyond the SQL as given by the initial CSS.

equal intensities $n_r = m_r \gg n_p = m_p$, we have

$$\kappa^2 = 2k^2 N n_p. \tag{15}$$

3.1.4. Decoherence from spontaneous scattering

According to Eq. (15) a doubling of the probe photon number doubles the κ^2 parameter. Hence one might expect increased squeezing and an improved angular definition of the the Bloch vector. However, this is not generally true since the dispersive coupling as characterized by Eq. (11) is inevitably accompanied by spontaneous photon scattering with a direct link provided by the Kramers-Kronig relations. If a fraction η of coherent atoms spontaneously scatters a photon, the length of the Bloch vector after light-atom interaction is $(1-\eta)|\langle\hat{\mathbf{J}}\rangle|$. Hence, for an initial CSS with fluctuations $\delta\hat{J}_z^{\mathrm{CSS}}$, the fluctuations of the resulting squeezed state needs to be less than $(1-\eta)\delta\hat{J}_z^{\mathrm{CSS}}$ in order to decrease the *angular* uncertainty of the Bloch vector in yz-plane [See Fig. 1c]. Thus, the variance of the squeezed state must be less than $(1-\eta)^2\delta^2\hat{J}_z^{\mathrm{CSS}}$ for *metrologically relevant* squeezing, i.e, $(1+\kappa^2)^{-1} < (1-\eta)^2$. With $\alpha = (\Gamma k/\Delta)(N/2)$ being the absorption coefficient for each probe and neglecting depletion, a total of $2n_p\alpha = n_p(\Gamma k/\Delta)N$ photons are spontaneously scattered and

$$\eta = \frac{\Gamma k}{\Delta} n_p. \tag{16}$$

The tradeoff between information gained via Eq. (15) and coherence lost via Eq. (16) in terms of number of probe photons is illustrated in Fig. 5. This leads to an optimal decoherence parameter η (*11*). We note that since (via k) *both* κ^2 and η are $\propto 1/\Delta^2$, the detuning enters the squeezing optimization completely equivalently to n_p. So despite the fact that the absorption falls off as $1/\Delta^2$ while the index of refraction only falls off as $1/\Delta$ the maximally achievable metrologically relevant squeezing for a given number of atoms N does not increase with the probe laser detuning: in principle, the maximum can be obtained for any detuning by adjusting the probe photon number accordingly. In practical experiments, to reduce the effect of probe frequency fluctuations a certain detuning may, however, be required (cf. section 3.1.2).

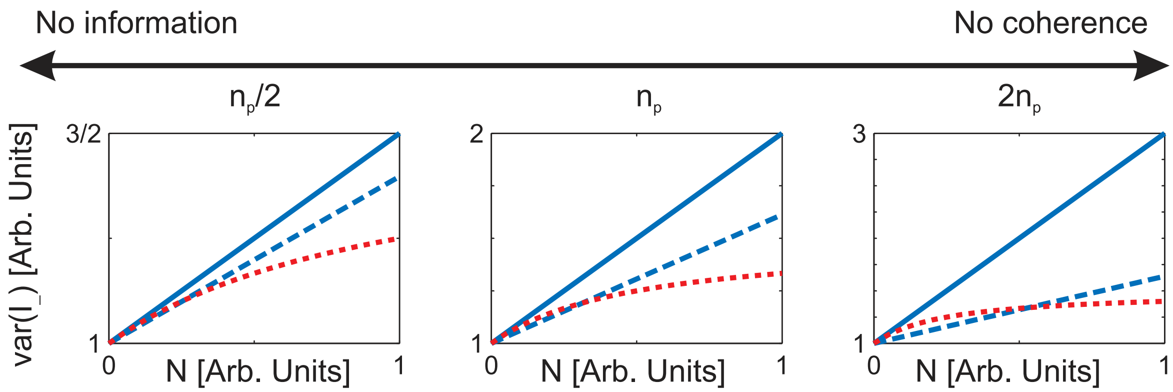

Figure 5. Squeezing and the effect of varying the number of probe photons. The full lines show the projection noise (variance) for a measurement on a CSS which scales linearly with atom number N. The outcome of a subsequent measurement can be predicted within the variance given by the dotted line. This conditionally reduced variance must be below the projection noise benchmark *scaled down* by a factor $(1-\eta)^2$ (dashed line) for the noise squeezing to be metrologically relevant [cf. Fig. 1(c)].

4. Conclusion

In conclusion, we have elaborated on some important perspectives of our recent experiments (*6*) demonstrating spectroscopically relevant quantum noise squeezing on the Cs clock transition. Here a conditionally squeezed state was produced by performing a QND measurement on the atomic pseudo-spin projection using a two-color dispersive probing scheme. The two-color protocol for which we have evaluated the noise terms benefits from several levels of common-mode rejection. We have provided a description of the atomic quantum state reduction resulting from the measurement on off-resonant probe light after QND interaction with a CSS. This generally gives rise to a state with reduced fluctuations of an atomic pseudo-spin component as compared to the initial CSS. We have discussed the conditions for this noise reduction to be of metrological relevance and aspects of the tradeoff between atomic decoherence and information gain.

Acknowledgements

This work was funded by the Danish National Research Foundation and the EU grants COMPAS, EMALI, and QAP. N.K. acknowledges support from the Danish Natural Science Research Council through a Steno Fellowship.

References

(1) Kuzmich, A.; Bigelow, N.P.; Mandel, L. *Europhys. Lett.* **1998**, *42*, 481–486.
(2) Fernholz, T.; Krauter, H.; Jensen, K.; Sherson, J.F.; Sørensen, A.S.; Polzik, E.S. *Phys. Rev. Lett.* **2008**, *101*, 073601.
(3) Teper, I.; Vrijsen, G.; Lee, J.; Kasevich, M.A. *Phys. Rev. A* **2008**, *78*, 051803.
(4) Takano, T.; Fuyama, M.; Namiki, R.; Takahashi, Y. *Phys. Rev. Lett.* **2009**, *102*, 033601.
(5) Schleier-Smith, M.H.; Leroux, I.D.; Vuletic, V., arXiv:0810.2582
(6) Appel, J.; Windpassinger, P.J.; Oblak, D.; Hoff, U.B.; Kjærgaard, N.; Polzik, E.S., arXiv:0810.3545.
(7) Dicke, R.H. *Phys. Rev.* **1954**, *93*, 99–110.
(8) Mandel, L.; Wolf, E. Optical Coherence and Quantum Optics; Cambridge University Press, 1997.
(9) Santarelli, G.; Laurent, P.; Lemonde, P.; Clairon, A.; Mann, A.G.; Chang, S.; Luiten, A.N.; Salomon, C. *Phys. Rev. Lett.* **1999**, *82*, 4619–4622.
(10) Wineland, D.J.; Bollinger, J.J.; Itano, W.M.; Moore, F.L.; Heinzen, D.J. *Phys. Rev. A* **1992**, *46*, R6797–R6800.
(11) Saffman, M.; Oblak, D.; Appel, J.; Polzik, E.S. *Phys. Rev. A* **2009**, *79*, 023831.
(12) Appel, J.; MacRae, A.; Lvovsky, A.I., arXiv:0809.3607

Appendix A. Phase detection

Referring to Fig. 2 and assuming a 50/50 beam splitter the differential detector output is

$$\begin{aligned}\hat{I}_- &= \left[\frac{1}{\sqrt{2}}(\hat{a}_p + \mathrm{i}\hat{a}_r)\right]^\dagger \left[\frac{1}{\sqrt{2}}(\hat{a}_p + \mathrm{i}\hat{a}_r)\right] - \left[\frac{1}{\sqrt{2}}(\mathrm{i}\hat{a}_p + \hat{a}_r)\right]^\dagger \left[\frac{1}{\sqrt{2}}(\mathrm{i}\hat{a}_p + \hat{a}_r)\right] \\ &= \mathrm{i}(\hat{a}_p^\dagger \hat{a}_r - \hat{a}_r^\dagger \hat{a}_p), \end{aligned} \tag{A1}$$

where $\hat{a}_p$ and $\hat{a}_r$ denote the annihilation operators for the probe and reference field, respectively. The reference and probe fields are described by coherent states of amplitudes α_r and α_p with mean photon numbers $n_r = \langle \hat{a}_r^\dagger \hat{a}_r \rangle = |\alpha_r|^2$ and $n_p = \langle \hat{a}_p^\dagger \hat{a}_p \rangle = |\alpha_p|^2$, respectively. Taking, without loss of generality, the coherent state amplitude for the reference field to be real $\alpha_r = \alpha_r^* = \sqrt{n_r}$ we obtain

$$\begin{aligned}\langle \hat{I}_- \rangle &= \langle \alpha_r, \alpha_p | \mathrm{i}(\hat{a}_p^\dagger \hat{a}_r - \hat{a}_r^\dagger \hat{a}_p) | \alpha_r, \alpha_p \rangle = \mathrm{i}(\alpha_p^* \alpha_r - \alpha_r^* \alpha_p) \\ &= 2\sqrt{n_r}\left(\frac{\alpha_p - \alpha_p^*}{2\mathrm{i}}\right) = 2\sqrt{n_r}\sqrt{n_p}\sin\phi, \end{aligned} \tag{A2}$$

where ϕ is the phase of the probe field with respect to the the reference field. Calculating the second moment of $\hat{I}_-$ we get

$$\begin{aligned}\langle \hat{I}_-^2 \rangle &= \langle \alpha_r, \alpha_p | \left[\mathrm{i}(\hat{a}_p^\dagger \hat{a}_r - \hat{a}_r^\dagger \hat{a}_p)\right]^2 | \alpha_r, \alpha_p \rangle \\ &= \langle \alpha_r, \alpha_p | \hat{a}_p \hat{a}_p^\dagger \hat{a}_r^\dagger \hat{a}_r + \hat{a}_p^\dagger \hat{a}_p \hat{a}_r \hat{a}_r^\dagger - \hat{a}_p^\dagger \hat{a}_p^\dagger \hat{a}_r \hat{a}_r - \hat{a}_p \hat{a}_p \hat{a}_r^\dagger \hat{a}_r^\dagger | \alpha_r, \alpha_p \rangle \\ &= (1 + n_p) n_r + n_p(1 + n_r) - 2n_r \frac{\alpha_p^{*2} + \alpha_p^2}{2} \\ &= n_r + n_p + 2n_r n_p(1 - \cos 2\phi) = n_r + n_p + 4n_r n_p \sin^2\phi. \end{aligned} \tag{A3}$$

Appendix B. Noise terms

In order to derive Eqns. (8) and (10) from Eq. (6) we must calculate var($\phi_\uparrow$) and var($\phi_\uparrow + \phi_\downarrow$), respectively.

B.1. *Single color probe*

Assuming $k_\uparrow$,$N_\uparrow$, ϕ_0 to be independent we obtain

$$\mathrm{var}\,(\phi_\uparrow) = \mathrm{var}\,(k_\uparrow N_\uparrow + \phi_0) = \mathrm{var}\,(k_\uparrow N_\uparrow) + \mathrm{var}\,(\phi_0)\,, \tag{B1}$$

$$\begin{aligned}\mathrm{var}\,(k_\uparrow N_\uparrow) &= \langle k_\uparrow \rangle^2 \underbrace{\mathrm{var}\,(N_\uparrow)}_{N/4} + \mathrm{var}\,(k_\uparrow) \underbrace{\langle N_\uparrow \rangle^2}_{N^2/4} + \mathrm{var}\,(k_\uparrow) \underbrace{\mathrm{var}\,(N_\uparrow)}_{N/4} \\ &= \left[\langle k_\uparrow \rangle^2 + \mathrm{var}\,(k_\uparrow)\, N + \mathrm{var}\,(k_\uparrow)\right] N/4, \end{aligned} \tag{B2}$$

where use of the CSS properties var($N_\uparrow$) $= N/4$ and $\langle N_\uparrow \rangle^2 = N^2/4$ has been made.

B.2. *Two color probe*

For $k \equiv k_\uparrow = k_\downarrow$ independent of $\Delta N \equiv N_\uparrow - N_\downarrow$

$$\begin{aligned}\mathrm{var}\left(\phi_\uparrow + \phi_\downarrow\right) = \mathrm{var}\left(k \Delta N\right) &= \langle k\rangle^2 \underbrace{\mathrm{var}\left(\Delta N\right)}_{N} + \mathrm{var}\left(k\right) \underbrace{\langle \Delta N\rangle^2}_{0} + \mathrm{var}\left(k\right) \underbrace{\mathrm{var}\left(\Delta N\right)}_{N} \\ &= \left[\langle k\rangle^2 + \mathrm{var}\left(k\right)\right] N, \end{aligned} \tag{B3}$$

using CSS properties $\mathrm{var}(N_\uparrow - N_\downarrow) = N$ and $\langle N_\uparrow - N_\downarrow\rangle^2 = 0$ (cf. section 2).